Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(02)00018-7
J. Appl. Maths Mechs, Vol. 66, No. 1, pp. 135–140, 2002

THE INDENTATION OF A FLAT PUNCH INTO A RIGID-PLASTIC HALF-SPACE†

R. I. NEPERSHIN
Moscow
e-mail: nepershin_r@pop.mtu.ru

(*Received* 28 *November* 2000)

The indentation of a flat punch into a rigid-plastic half-space is modelled by a centred field of slip lines with rotation of the rectilinear free boundary about the corner point of the punch. Adjacent to the rectilinear boundary, there is a rigid, stress-free region which is calculated using a velocity hodograph and determines the curvature of the initial horizontal boundary of the half-space during indentation up to the steady-state stage of the motion of the punch in the unbounded rigid-plastic medium.

The problem of the initial plane plastic flow of a rigid-plastic half-space with a rectilinear boundary during the indentation of a smooth punch [1, 3] has two solutions, Prandtl's and Hill's solutions, with the same limit pressure on the punch but a different kinematics of plastic flow. In Prandtl's solution it is assumed that a rigid region moves together with the punch and it is bounded by a straight slip line along which there is a discontinuity in the velocity while, in Hill's solution, it is assumed that there is slip of the plastic material along the contact boundary, which is possible in the case of an ideally smooth punch. Prandtl's solution is also applicable when there is contact friction which leads to the formation of a rigid region under the punch as in the case of a smooth punch.

Self-similar solutions of transient plastic flow problems in the case of the indentation of a smooth wedge [1] and the compression of a convex semi-bounded body by flat and curvilinear punches [3] are known. However, the Prandtl–Hill problem does not belong to the class of self-similar problems and, up to the present time, no analysis has been made of the indentation of a flat punch into a rigid-plastic medium. Calculations of the initial indentation of rigid punches into an elastoplastic half-space in the case of plane and axially symmetric strain using the finite-element method are known [4–6], but the severe distortion of the deformable boundary around the edge of the punch as a result of the singularity of the plastic flow in this region considerably restricts the possibility of such an analysis.

The statistically possible fields of slip lines of Prandtl's type have been used in technological problems for calculations of the pressure on a punch when it is indented into a half-space under the assumption that the rectilinear vertical plastic boundary around the punch is known [7, 8], but the process by which this boundary is formed was not considered.

In this paper, we consider a model of the transient process of the indentation of Prandtl's and Hill's punch into a rigid-plastic medium from the initial plastic flow up to the onset of the steady motion of the punch in an unbounded medium.

The unsteady indentation of a flat punch into a rigid-plastic half-space is modelled by the centred field of slip lines of Prandtl's and Hill's type with a discontinuous velocity field. Using a velocity hodograph, the relation between the displacement of the punch and the angle of rotation of the rectilinear stress-free boundary is obtained and the curvilinear boundary of the half-space is found. At the beginning of the indentation, the boundary of the half-space which is displaced has a horizontal segment and two curvilinear segments formed by the material which leaves the plastic region. After the disappearance of the horizontal segment, the curvilinear boundary asymptotically approaches the central axis of the punch, forming a deep crater at the instant when the transition to steady plastic flow occurs.

At the stage of a steady flow over Prandtl's punch, a hollow is formed with a straight stress-free boundary, reaching the vertical axis of symmetry and coinciding with the streamline of the velocity field. At the stage of the steady motion of Hill's punch in a rigid-plastic medium, the hollow under the punch disappears. In this case, we obtain a flow about an infinitely thin plate, the lower boundary of which is loaded by the plastic pressure while the upper boundary coincides with the stress-free steady boundary

†*Prikl. Mat. Mekh.* Vol. 66, No. 1, pp. 140–146, 2002.

of the plastic flow. For both punches, a dependence of the pressure on the punch on the depth of the indentation is obtained which asymptotically approaches the maximum value for the steady flow.

1. A MODEL OF THE PLASTIC FLOW

When a flat punch is impressed indented along a normal to the boundary of a rigid-plastic half-space, the plastic flow region has a vertical plane of symmetry passing through the punch centre, which we shall take as the origin of a Cartesian system of coordinates. We shall consider the right-hand half of the plastic region. We will assume that the coordinates (x, y), the stresses and the velocities are dimensionless, taking the half-width of the punch, twice the plastic constant $2k$ (k is the shear yield stress) and the rate of motion of the punch as the units of length, stress and velocity respectively.

The field of slip lines and velocity hodographs which satisfy the static and kinematic boundary conditions of the problem and the condition for the irreversibility of the dissipation of the energy of plastic flow are shown in Fig. 1 for Prandtl's punch and in Fig. 2 for Hill's punch. In Prandtl's and Hill's solutions, the region ACD is loaded up to the plastic state by compression along the horizontal stress free boundary AD. A rigid wedge with a vertex at the point D is also loaded up to the plastic state by compression along the boundary of the half-space. The pressure on the punch, which corresponds to the plastic loading of the region ACD, is equal to the known value

$$p = 1 + \pi/2 \tag{1.1}$$

When the punch is indented, the boundary AD rises above the boundary of the half-space and the supporting power of the rigid wedge in the neighbourhood of point D is insufficient to transmit the plastic compressive stress from the region ACD. As a result, unloading of the material occurs in the region ACD and in the rigid region to the right of the boundary CD. We will assume that the rigid plastic boundary CD is loaded by pure shear (Fig. 3). The region ACD is loaded by a uni-axial compression $\sigma_2 = -1/2$ along the boundary AD, and the region A^*CD by a uniaxial stress $\sigma_1 = +1/2$ along the boundary A^*D.

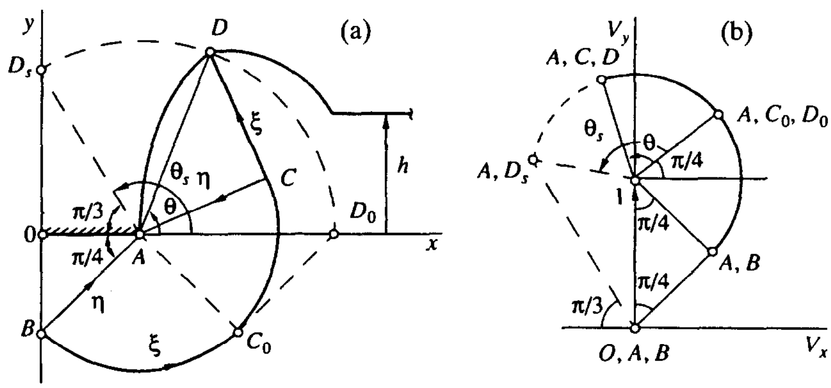

Fig. 1

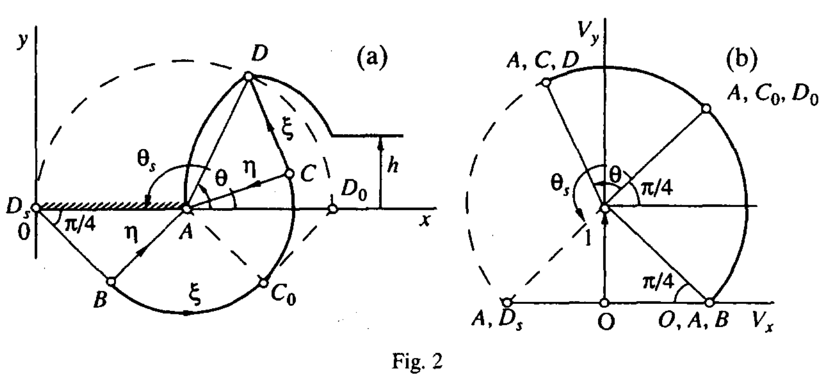

Fig. 2

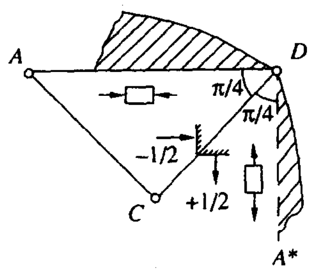

Fig. 3

The Mohr circles for pure plastic shear along the glide line CD and uniaxial loading in the above-mentioned regions are shown in Fig. 4. The rigid regions in front of the boundaries AD and A^*D are stress-free. The points P and P_1 are the common poles for plastic shear along the slip line CD and uniaxial loading in the regions ACD and A^*CD with continuous normal stresses in the planes perpendicular to AD and A^*D. The mean stress σ changes abruptly along the normal to the slip line CD, taking a zero value on this slip line.

Hence, in the case of pure shear along the isolated slip line CD, the condition that there is no plastic flow in the rigid zones is satisfied for the slip line fields shown in Figs 1 and 2. The angle of inclination θ of the boundary AD changes from zero for the initial position of the punch to the steady value $\theta_s = 2\pi/3$ in the case of Prandtl's punch and $\theta_s = \pi$ for Hill's punch. In the case of a steady flow, the boundary AD reaches the vertical line of symmetry and coincides with the streamline. A closed cavity with an inclined stress-free boundary AD is formed above Prandtl's punch. In the case of Hill's punch, the boundary AD coincides with the upper boundary of the punch $y = +0$, the lower boundary of which $y = -0$ is loaded with the plastic pressure.

The slip line fields and Hencky's integrals determine the normal pressure on the punch as a function of the angle θ

$$p = (1+\pi)/2 + \theta, \qquad 0 \leqslant \theta \leqslant \theta_s \tag{1.2}$$

The limit pressures on the punch in the case of steady flow are equal to $(3 + 7\pi)/6$ for Prandtl's punch and $(1 + 3\pi)/2$ for Hill's punch.

The velocity hodographs are shown in Fig. 1(b) for Prandtl's punch and in Fig. 2(b) for Hill's punch. We impose a vertical velocity on the system which is the reverse of the velocity of the punch. In this case, the punch is fixed and the rigid-plastic medium moves with a velocity $V = 1$. The hodograph for Prandtl's punch is defined by the two different discontinuities in the velocity $[V] = 1/\sqrt{2}$ which arise at point B and propagate along the rigid-plastic boundaries BA and BD of the slip line field. The hodograph for Hill's punch is defined by a single discontinuity in the velocity $[V] = \sqrt{2}$. The velocity field of the region ABC of the physical plane is mapped onto the hodograph as an arc of a circle with centre at the point (1, 0). The velocity vector of the points in the region ACD is a function of the angle θ and, for both punches, its components can be written in the form

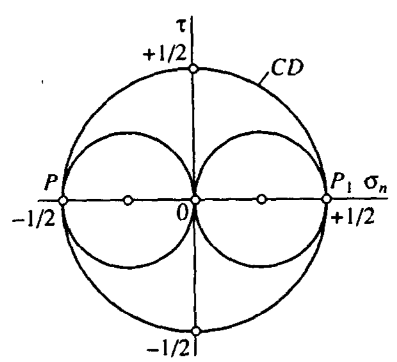

Fig. 4

$$V_x = C\zeta_-(\theta), \quad V_y = 1 + C\zeta_+(\theta); \quad \zeta_\pm(\theta) = (\cos\theta \pm \sin\theta) \tag{1.3}$$

$$C = \begin{cases} \frac{1}{2} & \text{for Prandtl's punch} \\ 1 & \text{for Hill's punch} \end{cases}$$

2. THE BOUNDARY OF THE HALF-SPACE

When the angle θ increases from zero to θ_s, the boundary AD of the region ACD rotates about the fixed corner point A, and the point D, which reaches the stress-free external boundary of the half-space, describes an arc of the circle $D_0 - D_s$ (Fig. 1), the equation of which is

$$x = 1 + \cos\theta / C, \qquad y = \sin\theta / C \tag{2.1}$$

Point D is common to the left-hand stress-free rigid region, adjacent to the boundary AD, and to the right-hand rigid region which moves at a velocity $V_y = 1, V_x = 0$. When the punch is indented, the velocity normal to the boundary AD is positive and material particles cross the boundary AD, forming a stress-free rigid region in front of this boundary. Simultaneously with this, as a consequence of the rotation of the region ACD, material particles cross the boundary DC and reach the right-hand rigid region. Since the velocity of the point D in the region ACD is greater than the velocity of the right-hand rigid region, a falling curvilinear boundary is formed to the right of the point D, which passes into the horizontal boundary of the half-space.

The displacement h of the boundary of the half-space relative to the punch and the shape of the rigid regions are functions of the angle θ. At the beginning of the indentation, the horizontal boundary around the punch rises as a rigid body above the remaining boundary of the half-plane, forming a flat part. To the left of this part, an ascending curvilinear boundary AF appears, which is formed by material particles leaving the plastic region at the corner point of the punch a. Point D is the right-hand boundary of the flat part. When the angle θ increases from zero to the value θ^*, the flat part of the raised boundary disappears and the whole of the boundary of the half-space around the punch becomes curvilinear.

The first stage of indentation. $\theta \leqslant \theta$. When there is a flat part of the raised boundary, the coordinates y of this boundary and the point D are defined by the second equation of (2.1), from which we find the increment $dy = (\cos\theta/C)d\theta$. The increments dh and dy are proportional to the velocities V_y of the point D of the region ACD and $V = 1$ of the right-hand rigid region $dh = dy/V_y$, where V_y is defined by the second equation of (1.3). The relation between dh and $d\theta$ follows from this, and, after integration, we find

$$h(\theta) = \int_0^\theta \frac{\cos\theta}{C[1 + C\zeta_+(\theta)]} d\theta, \qquad \theta \leqslant \theta^* \tag{2.2}$$

Integral (2.2) defines the vertical displacement of the horizontal boundary of the half-space when $x \geqslant 1 + 1/C$. The coordinates x and y of the curvilinear boundary $D - D_0$ are defined by Eqs (2.1) for values of $\theta_1 \leqslant \theta$ and by the vertical displacement $h_1(\theta, \theta_1)$, that is, integral (2.2) with a lower limit of θ_1

$$x = 1 + \cos\theta_1 / C, \qquad y = \sin\theta_1 / C + h_1(\theta, \theta_1), \qquad 0 \leqslant \theta_1 \leqslant \theta \tag{2.3}$$

The increments of the coordinates dx and dy of material particles, moving from the corner point a and forming the left-hand curvilinear boundary AF, are proportional to the velocities (1.3) with a coefficient of proportionality dh. After integration using relations (2.1), we find the x and y coordinates of the boundary AF

$$x = 1 + \int_0^{\theta_1} \frac{\cos\theta\zeta_-(\theta)}{1 + C\zeta_\pm(\theta)} d\theta, \qquad y = \frac{\sin\theta_1}{C}, \qquad 0 \leqslant \theta_1 \leqslant \theta \tag{2.4}$$

The coordinates of the point F are defined by Eqs (2.4) when $\theta_1 = \theta$. We find the final value of the angle θ^* of the first stage of the indentation of the punch from Eqs (2.1) and (2.4), by equating the x coordinates of the points D and F

$$\cos\theta^* = C\int_0^{\theta^*} \frac{\cos\theta(\cos\theta - \sin\theta)}{1 + C(\cos\theta + \sin\theta)} d\theta \tag{2.5}$$

The values $\theta^* = 1.479$ for Prandtl's punch and $\theta^* = 1.434$ for Hill's punch were obtained by the numerical solution of Eq. (2.5) using Newton's method. The boundaries of the half-space for the first stage of the indentation of Prandtl's punch (the continuous curve, $p = 3.09$) and Hill's punch (the dashed curve, $p = 2.99$) are shown in Fig. 5(a).

The second stage of indentation. $\theta^* < \theta < \theta_s$. In the second stage of the indentation of the punch, when the angle θ is increased by $d\theta$, the line AD crosses the curvilinear boundary of the left-hand rigid domain, which corresponds to the angle θ. Material particles in the neighbourhood of point D pass from the left-hand stress-free rigid region into the region ACD and leave it through the boundary DC, reaching the right-hand rigid region. We find the point P of the left-hand rigid region, which reaches a new point D for an increment of the angle $d\theta$, at the intersection of the line, which passes through the point D parallel to the velocity vector (1.3), with the curvilinear boundary AD of the left-hand rigid region corresponding to the angle θ.

For this purpose, we will approximate the curvilinear boundary of the region to the left of the line AD using a piecewise-linear relation. We find the direction of the mean velocity vector of the points of the boundary AD when the angle θ is increased by $d\theta$ from relations (1.3)

$$\mathrm{tg}\,\alpha = V_y(\theta + d\theta/2)/V_x(\theta + d\theta/2)$$

We approximate the direction of the tangent to the boundary of the rigid region close to the point D by the angle of inclination of the chord between the points (x_j, y_j) and (x_{j+1}, y_{j+1}) of this boundary

$$\mathrm{tg}\,\beta = (y_{j+1} - y_j)/(x_{j+1} - x_j)$$

We denote the coordinates of the new point D, which are determined by expressions (2.1) for an angle $\theta = d\theta$, by x_D and y_D and find the point of intersection of the straight line passing through the point D with an angle of inclination α and the straight line passing through the points (x_j, y_j) and (x_{j+1}, y_{j+1}) with an angle of inclination β

$$x = (y_D - y_j + x_j\,\mathrm{tg}\,\beta -$$
$$-x_D\,\mathrm{tg}\,\alpha)/(\mathrm{tg}\,\beta - \mathrm{tg}\,\alpha)$$
$$y = y_j + (x - x_j)\,\mathrm{tg}\,\beta \tag{2.6}$$

If the y coordinate satisfies the inequality

$$y_{j+1} < y < y_j$$

then expressions (2.6) define the point P of the boundary of the rigid region, which is shifted to the point D of the new rigid region. Otherwise, we reduce the subscripts j until the above-mentioned inequality is satisfied.

The vector for the displacement of the boundary AD of the left-hand rigid region in the new position is defined by the increments of the coordinates

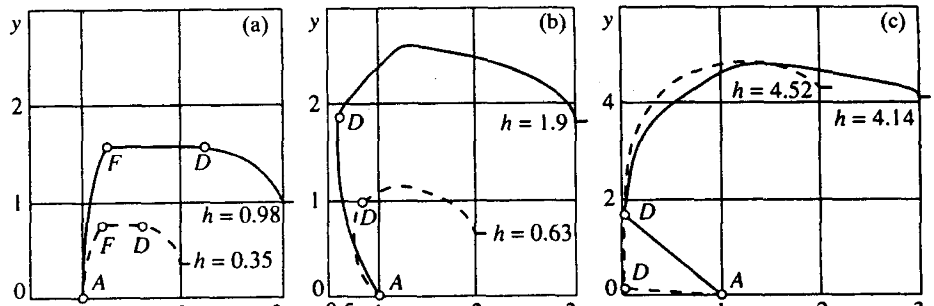

Fig. 5

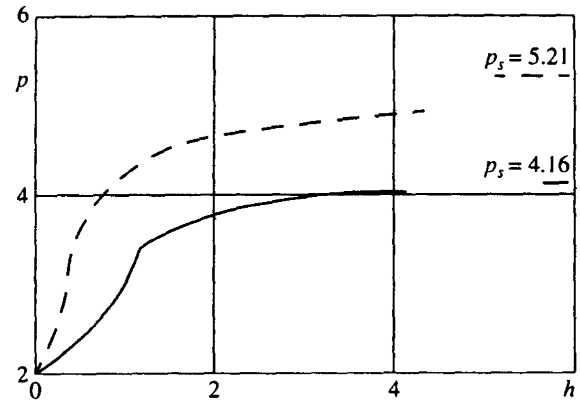

Fig. 6

$$dx = 1 + \cos(\theta + d\theta) / C - x_P, \quad dy = \sin(\theta + d\theta) / C - y_P \tag{2.7}$$

and the vertical displacement of the boundary of the right-hand rigid region is defined by the equation

$$dh = dy / [1 + C(\sin\theta + \cos\theta)] \tag{2.8}$$

In the transient stage of the indentation of a punch $\beta \neq \alpha$ and expressions (2.6) uniquely define the point P. The equality $\beta = \alpha$ denotes the beginning of steady plastic flow for which the boundary AD coincides with the direction of the velocity vector of the points of this boundary, and the point P cannot be found using expressions (2.6). Hence, the beginning of the steady stage of the flow around a punch can only be determined approximately for values of θ which are close to θ.

The boundaries of the half-space for the second stage of the indentation of Prandtl's punch (the continuous curve, $p = 3.85$) and Hill's punch (the dashed curve, $p = 3.85$) are shown in Fig. 5(b). In the second stage of indentation, the direction of the velocity vector (1.3) of the points of the region ACD approaches the direction of the line AD in the physical plane. The displacement vector of the point P and the increments dh rapidly increase. The stress-free rigid region to the left of the rectilinear boundary AD disappears and the boundary of the right-hand rigid region above the point D assumes the form of a deep crater, compared with the half-width of the punch, which approaches the axis $x = 0$ asymptotically when $\theta = \theta_s$. The boundaries of the half-space on transforms to steady plastic flow are shown in Fig. 5(c) for Prandtl's punch (the continuous curve, $p = 4.14$) and Hill's punch (the dashed curve, $p = 5.0$). The corresponding approximate values of the depth of indentation at the beginning of the steady flow are equal to $h \approx 5$ for Prandtl's punch and $h \approx 7.5$ for Hill's punch.

A program was written for calculating the curvilinear boundaries of the half-space during the indentation of Prandtl's and Hill's punches. The accumulated relative error of the integral condition for the incompressibility of the material did not exceed 0.005 for an angular step $d\theta = 0.01\theta$ at all stages of indentation.

Graphs of the pressure on the punch against the depth of indentation are shown in Fig. 6. These asymptotically approach the values $(3 + 7\pi)/6$ and $(1 + 3\pi)/2$ for steady flow around Prandtl's punch (the continuous curve) and Hill's punch (the dashed curve).

REFERENCES

1. HILL, R., *The Mathematical Theory of Plasticity*. Clarendon Press, Oxford, 1985.
2. IVLEV, D. D., *The Theory of Ideal Plasticity*. Nauka, Moscow, 1966.
3. BYKOVTSEV, G. I. and IVLEV, D. D., *The Theory of Plasticity*. Dal'nauka, Vladivostok, 1998.
4. LEE, C. H. and KOBAYASHI, S., Elastoplastic analysis of plane-strain and axisymmetric flat punch indentation by the finite-element method. *Int. J. Mech. Sci.* 1970, **12**, 349–370.
5. HARDY, C., BARONET, C. N. and TORDION, G. V., The elasto-plastic indentation of a half-space by a rigid sphere. *Int. J. Numr. Meth. Engng*, 1971, **3**, 451–462.
6. VOYIADJIS, G. Z. and BUCKNER, N. E., Indentation of a half-space with a rigid indentor. *Int. J. Numr. Meth. Engng*, 1983, **19**, 1555–1578.
7. TOMLENOV, A. D., *The Theory of the Plastic Deformation of Metals*. Metallurgiya, Moscow, 1972.
8. JOHNSON, W. and MELLOR, P. B., *Engineering Plasticity*. Van Nostrand Reinhold, Leningrad 1975.

Translated by E.L.S.